# Combining Classical Logic and Intuitionistic Logic

## Double Negation, Polarization, Focusing, and Semantics

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# Challenges of Combining Intuitionistic Logic with Classical Logic Using a Double Negation Translation

- Intuitionistic implication should not collapse into classical implication. Consider $A \vee^e (B \supset C) \longrightarrow \sim(\sim A \wedge \sim(B \supset C))$
- How to distinguish the introduction of a translated classical "connective" from intuitionistic introductions.

  Introduction of classical disjunction $A \vee^e B$:

  $$\dfrac{\dfrac{\sim A, \sim B, \Gamma \vdash}{\sim A \wedge \sim B, \Gamma \vdash}}{\Gamma \vdash \sim(\sim A \wedge \sim B)}$$

  No guarantee that sequence won't be interrupted by other rules.
- How to recognize classical "dualities". How is $\sim(\sim A \wedge \sim B)$ the "dual" of $(\sim A \wedge \sim B)$ in an intuitionistic sense, given that $\sim\sim P \not\equiv P$.

# Challenges Continued ...

- How to distinguish classical from intuitionistic *equivalence.*
- Classical versus Intuitionistic **Cut Elimination**.

$$\frac{A, \Gamma \vdash B \quad \sim A, \Gamma \vdash B}{\Gamma \vdash B}$$

**Admissible in classical logic but not in intuitionistic logic.**

*How do we simulate this cut in an intuitionistic proof system?*

- What is the *meaning* of mixed formulas such as $A \vee^e (B \supset (C \vee D))$?.

# Outline and Overview:

- **Goal: Combine LJ and LC into *Polarized Intuitionistic Logic***
- **LC does not include intuitionistic implication**
- Start with Intuitionistic Logic with a designated atom $\bot$.
- $\bot$ is just minimal ”false” - this logic predates ICL.
  - ICL is a stand-alone logic
  - PIL **combine** logics
- Assign labels, i.e., ”polarities” to formulas.
- Define Double-Negation translation.
- Use focusing (focalization) to isolate ”classical connectives”
- Derive Unified Sequent Calculus
- Define Kripke/Algebraic Semantics

# Syntax and Colors

- Formulas freely generated from atoms, $\wedge$, $\vee$, $\supset$, 0 and designated atom $\perp$.
- Define $\neg A \ = \ A \supset \perp$; $(A \supset 0 \ = \ \sim A)$
- Formulas are **Red** or **Green** as follows:
  - $A \wedge B$, $A \vee B$, 0, and $A \supset B$ where $B \neq \perp$ are red.
  - All atoms are red, except $\perp$, which is green.
  - $\neg A \ \ (A \supset \perp)$ is green.
    - $\neg^{2n}(R)$, $n > 0$, are *reddish green* (also includes $\perp$)
    - $\neg^{2n+1}(R)$ are *solidly green*
- Red and Green formulas can be logically equivalent:
  $(A \wedge B) \supset \perp \ \equiv \ A \supset (B \supset \perp)$
- This polarization is not same as duality in linear logic: $?X \multimap !Y$

# Recovering Classical "Dualities"

- $M^{\perp} \;=\; \neg M$ for **red or reddish-green** $M$
- $(\neg M)^{\perp} \;=\; M$

**Syntactic Identity:** $A^{\perp\perp} = A$

- $A^{\perp}$ is convenient way to refer to doubly-negated formulas
- $A^{\perp}$ is not a connective.
- if $A \equiv B$, then $A^{\perp}$ is only *classically* $\equiv$ to $B^{\perp}$
  $((A \wedge B) \supset \bot)^{\perp} \;=\; A \wedge B, \quad (A \supset (B \supset \bot))^{\perp} \;=\; \neg(A \supset \neg B)$

# Double Negation as Macro Expansion

$R$ **red** and $E$ **green**

- $A \vee^e B \;=\; (A^\perp \wedge B^\perp)^\perp \;=\; \neg(A^\perp \wedge B^\perp)$
- $A \wedge^e B \;=\; (A^\perp \vee B^\perp)^\perp \;=\; \neg(A^\perp \vee B^\perp)$
- $1 \;=\; \perp^\perp \;=\; \perp \supset \perp; \quad \top \;=\; 0^\perp \;=\; 0 \supset \perp$

**To complete the definition of $A^\perp$, we need *missing link:***

- $A \propto B \;=\; \neg(A \supset B^\perp)$
  includes special case: $(R \propto 1) = \neg\neg R$

**These are not *yet* new connectives, just labels**

**The following holds:**

- $1^{\perp} \ = \ \perp; \quad \top^{\perp} \ = \ 0$
- $(A \vee^e B)^{\perp} \ = \ A^{\perp} \wedge B^{\perp}$
- $(A \wedge^e B)^{\perp} \ = \ A^{\perp} \vee B^{\perp}$
- $(A \propto B)^{\perp} \equiv A \supset B^{\perp} \qquad (\text{mod } \neg\neg\neg P \equiv \neg P)$

**Caution: do not equate "green" with "classical."**

**Classical fragment will use $\vee$ and $\vee^e$, $\wedge$ and $\wedge^e$, $0$ and $\perp$, $1$ and $\top$.**

**The classical fragment will be more LC than LK.**

# Intuitionistic Sequent Calculus LJ

$$\frac{A, B, \Gamma \vdash D}{A \wedge B, \Gamma \vdash D} \wedge L \qquad \frac{A, \Gamma \vdash D \quad B, \Gamma \vdash D}{A \vee B, \Gamma \vdash D} \vee L \qquad \frac{A \supset B, \Gamma \vdash A \quad B, \Gamma \vdash D}{A \supset B, \Gamma \vdash D} \supset L$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge R \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2} \vee R \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B} \supset R$$

$$\frac{}{a, \Gamma \vdash a}\ Id \qquad \frac{}{0, \Gamma \vdash D}\ 0L \qquad \frac{}{\Gamma \vdash 1}\ 1R$$

$\bot$ is considered a special atom

# $\vee^e$, $\wedge^e$ and $\propto$ as Synthetic Connectives in LJ?

**Distinguish between sequents $\Gamma \vdash A$ and $\Gamma \vdash \bot$:**

**Correspond to sequents with and without a *stoup***

**Introduction of a green formula $\neg A = A \supset \bot$:**

$$\frac{A, \Gamma \vdash \bot}{\Gamma \vdash A \supset \bot} \supset R \qquad \frac{A \supset \bot, \Gamma \vdash A \quad \dfrac{}{\bot, \Gamma \vdash \bot}\, Id}{A \supset \bot, \Gamma \vdash \bot} \supset L$$

**But LJ is not good enough**

We don't want the following:

$$\frac{B \supset C, \Gamma \vdash B \quad C, \Gamma \vdash A \supset \bot}{B \supset C, \Gamma \vdash A \supset \bot} \qquad \frac{A \supset \bot, \Gamma \vdash A \quad \bot, \Gamma \vdash B}{A \supset \bot, \Gamma \vdash B}$$

# Looking for one-to-one mapping between proofs

**Derive new introduction rules for $A \vee^e B = \neg(A^\perp \wedge B^\perp)$:**

$$\frac{\Gamma \vdash_\bullet A, B}{\Gamma \vdash A \vee^e B}\ \vee^e R \qquad \frac{A \vee^e B, \Gamma \vdash_\circ A^\perp \quad A \vee^e B, \Gamma \vdash_\circ B^\perp}{A \vee^e B, \Gamma \vdash_\bullet}\ \vee^e L$$

Want this to correspond one-to-one with the following fragments:

$$\cfrac{\cfrac{A^\perp, B^\perp, \Gamma \vdash \perp}{A^\perp \wedge B^\perp, \Gamma \vdash \perp}}{\Gamma \vdash \neg(A^\perp \wedge B^\perp)} \qquad \cfrac{\cfrac{\neg(A^\perp \wedge B^\perp), \Gamma \vdash A^\perp \quad \neg(A^\perp \wedge B^\perp), \Gamma \vdash B^\perp}{\neg(A^\perp \wedge B^\perp), \Gamma \vdash A^\perp \wedge B^\perp} \qquad \perp, \Gamma \vdash \perp}{\neg(A^\perp \wedge B^\perp), \Gamma \vdash \perp}$$

**Need focused intuitionistic sequent calculus (LJF)**

**even for *unfocused* synthetic introduction rules**

# A new dimension of polarization

- Atoms are **"positive,"** except $\bot$, which is **"negative"**
- $\vee$, $\wedge^+$, 1 and 0 are positive
- $\wedge^-$, $\supset$, are negative
- Positives are "synchronous" on the right; Negatives are synchronous on the left
- Asynchronous rules are always invertible
- Synchronous (and asynchronous) rules can be stringed together into a single phase.
- $A \vee^e B \;=\; (A^\bot \wedge^+ B^\bot)^\bot$
- **Caution:** Do not confuse **positive** with **red** polarities:
  $A \supset B$ is **red** but **negative** (red=positive only in LC)

# Use Delays to Fine-Tune Focusing

$$\partial^+(A) \;=\; A \wedge^+ 1; \qquad \partial^-(A) \;=\; 1 \supset A$$

| $F$ | $F^\ell$ (left) | $F^r$ (right) |
|---|---|---|
| atomic $a$ | $\partial^-(a)$ | $a$ |
| $0$ | $\partial^-(0)$ | $0$ |
| $1$ | $\partial^-(1)$ | $1$ |
| $A \wedge B$ | $\partial^+(A^\ell) \wedge^- \partial^+(B^\ell)$ | $\partial^+(A^r \wedge^- B^r)$ |
| $A \vee B$ | $\partial^-(A^\ell \vee B^\ell)$ | $\partial^-(A^r) \vee \partial^-(B^r)$ |
| $A \supset B$ | $\partial^-(A^r) \supset \partial^+(B^\ell)$ | $\partial^+(A^\ell \supset B^r)$ |

| $F$ | $F^\ell$ (left) | $F^r$ (right) |
|---|---|---|
| $\perp$ | $\perp$ | $\perp$ |
| $a^\perp$, atomic $a$ | $\neg(\partial^-(a))$ | $\neg\partial^-(a)$ |
| $A \wedge^e B$ | $\neg(\partial^-(A^{\perp r}) \vee \partial^-(B^{\perp r}))$ | $\neg\partial^-(A^{\perp\ell} \vee B^{\perp\ell})$ |
| $A \vee^e B$ | $\neg(\partial^-(A^{\perp r}) \wedge^+ \partial^-(B^{\perp r}))$ | $\neg\partial^-(A^{\perp\ell} \wedge^+ B^{\perp\ell})$ |
| $A \propto B$ | $\neg(A^\ell \supset B^{\perp r})$ | $\neg(\partial^-(A^r) \supset \partial^+(B^{\perp\ell}))$ |

# Deriving the Sequent Calculus *LP*

**Different modes of sequents:**

- $\Gamma \vdash_{\bullet} A_1, \ldots, A_n \quad \cong \quad A_1^{\perp}, \ldots, A_n^{\perp}, \Gamma \vdash \perp \qquad (\Gamma \vdash_{\bullet} \;\cong\; \Gamma \vdash \perp)$
- $\Gamma \vdash_{\circ} A \quad \cong \quad \Gamma \vdash A$

**Structural Rules** ($R$ red, $E$ green)

$$\frac{\Gamma \vdash_{\bullet} E}{\Gamma \vdash_{\circ} E}\ \mathit{Signal/Stop} \qquad\qquad \frac{A^{\perp}, \Gamma \vdash_{\circ} A}{A^{\perp}, \Gamma \vdash_{\bullet}}\ \mathit{Load/Go}$$

$$\cong$$

$$\cfrac{\cfrac{\cfrac{[\Gamma, \partial^{-}(A)] \longrightarrow [\perp]}{[\Gamma, \partial^{-}(A)] \longrightarrow \perp}}{[\Gamma], \partial^{-}(A) \longrightarrow \perp}}{[\Gamma] \longrightarrow \partial^{-}(A) \supset \perp} \qquad \cfrac{\cfrac{\cfrac{[A \supset \perp, \Gamma] \longrightarrow A}{[A \supset \perp, \Gamma] -_{A}\!\!\rightarrow} \qquad \cfrac{}{[A \supset \perp, \Gamma] \xrightarrow{\perp} [\perp]}}{[A \supset \perp, \Gamma] \xrightarrow{A \supset \perp} [\perp]}}{[A \supset \perp, \Gamma] \longrightarrow [\perp]}$$

$$\dfrac{\Gamma \vdash_\circ A \quad \Gamma \vdash_\bullet B}{\Gamma \vdash_\bullet A \propto B} \propto R$$

$$\downarrow$$

$$\dfrac{\dfrac{\dfrac{[\ldots,\Gamma] \longrightarrow \partial^-(A^r)}{[\ldots,\Gamma] -_{\partial^-(A^r)}\!\!\longrightarrow} R_r \qquad \dfrac{[\ldots,\Gamma], \partial^+(B^{\perp\ell}) \longrightarrow [\perp]}{[\ldots,\Gamma] \overset{\partial^+(B^{\perp\ell})}{\longrightarrow} [\perp]} R_\ell}{[\ldots,\Gamma] \overset{\partial^-(A^r) \supset \partial^+(B^{\perp\ell})}{\longrightarrow} [\perp]} \supset L}{[\partial^-(A^r) \supset \partial^+(B^{\perp\ell}), \Gamma] \longrightarrow [\perp]} Lf$$

**Correspondence between focusing phases and synthetic introduction rules must be relaxed:**

$A \propto B \equiv (A \supset B^\perp) \supset \perp$, which is $-$ followed by $+$

$-+$, $+-$ are OK, but not $+-+$.

# Sequent Calculus LP

## Structural Rules and Identity

$$\frac{\Gamma \vdash_\bullet E}{\Gamma \vdash_\circ E}\ \textit{Signal} \qquad \frac{A, \Gamma \vdash_\bullet \Theta}{\Gamma \vdash_\bullet A^\perp, \Theta}\ \textit{Store} \qquad \frac{A^\perp, \Gamma \vdash_\circ A}{A^\perp, \Gamma \vdash_\bullet}\ \textit{Load} \qquad \frac{}{a, \Gamma \vdash_\circ a}\ I$$

## Right-Red Introduction Rules

$$\frac{\Gamma \vdash_\circ A \quad \Gamma \vdash_\circ B}{\Gamma \vdash_\circ A \wedge B}\ \wedge R \qquad \frac{\Gamma \vdash_\circ A_i}{\Gamma \vdash_\circ A_1 \vee A_2}\ \vee R \qquad \frac{A, \Gamma \vdash_\circ B}{\Gamma \vdash_\circ A \supset B}\ \supset R$$

## Left-Red Introduction Rules

$$\frac{A, B, \Gamma \vdash_\circ R}{A \wedge B, \Gamma \vdash_\circ R}\ \wedge L \qquad \frac{A, \Gamma \vdash_\circ R \quad B, \Gamma \vdash_\circ R}{A \vee B, \Gamma \vdash_\circ R}\ \vee L \qquad \frac{A \supset B, \Gamma \vdash_\circ A \quad B, \Gamma \vdash_\circ R}{A \supset B, \Gamma \vdash_\circ R}\ \supset L$$

## Right-Green Introduction Rules

$$\frac{\Gamma \vdash_\bullet A \quad \Gamma \vdash_\bullet B}{\Gamma \vdash_\bullet A \wedge^e B}\ \wedge^e R \qquad \frac{\Gamma \vdash_\bullet A, B}{\Gamma \vdash_\bullet A \vee^e B}\ \vee^e R \qquad \frac{\Gamma \vdash_\circ A \quad \Gamma \vdash_\bullet B}{\Gamma \vdash_\bullet A \propto B}\ \propto R$$

## Rules for Constants

$$\frac{}{\Gamma \vdash_\circ 1}\ 1R \qquad \frac{\Gamma \vdash_\circ R}{1, \Gamma \vdash_\circ R}\ 1L \qquad \frac{}{0, \Gamma \vdash_\circ R}\ 0L \qquad \frac{\Gamma \vdash_\bullet}{\Gamma \vdash_\bullet \bot}\ \bot R \qquad \frac{}{\Gamma \vdash_\bullet \top}\ \top R$$

# Extends to First Order

**Rules for Quantifiers**

$$\frac{\Gamma \vdash_\circ A[t/x]}{\Gamma \vdash_\circ \exists x.A}\ \exists R \qquad \frac{\Gamma \vdash_\circ A}{\Gamma \vdash_\circ \Pi y.A}\ \Pi R \qquad \frac{A, \Gamma \vdash_\circ R}{\exists y.A, \Gamma \vdash_\circ R}\ \exists L \qquad \frac{A[t/x], \Pi x.A, \Gamma \vdash_\circ R}{\Pi x.A, \Gamma \vdash_\circ R}\ \Pi L$$

$$\frac{\Gamma \vdash_\bullet A[t/x]}{\Gamma \vdash_\bullet \Sigma x.A}\ \Sigma R \qquad \frac{\Gamma \vdash_\bullet A}{\Gamma \vdash_\bullet \forall y.A}\ \forall R$$

Here, $y$ is not free in $\Gamma$ and $R$.

**Why not remove delays and get focused *LPF*?**

**Possible, but first ...**

# LC Inside LP

$$\frac{\vdash \Gamma, N, P; S}{\vdash \Gamma, N \vee P; S} \quad \longmapsto \quad \frac{\Gamma, P, N \vdash_{\circ} S}{\Gamma, P \wedge N \vdash_{\circ} S} \wedge L$$

$$\frac{\vdash \Gamma, N, P;}{\vdash \Gamma, N \vee P;} \quad \longmapsto \quad \frac{\Gamma \vdash_{\bullet} N, P}{\Gamma \vdash_{\bullet} N \vee^{e} P} \vee^{e} R$$

$$\frac{\vdash \Gamma; P \quad \vdash \Delta, N;}{\vdash \Gamma\Delta; P \wedge N} \quad \longmapsto \quad \frac{\Gamma \vdash_{\circ} P \quad \dfrac{\Gamma \vdash_{\bullet} N}{\Gamma \vdash_{\circ} N} \mathit{Signal}}{\Gamma \vdash_{\circ} P \wedge N} \wedge R$$

**LC invariant: no "positive" introductions outside of the stoup**

**...subsumed by LP invariant: *no green introduction in $\vdash_{\circ}$ mode***

**LC is accidentally almost focused, but not LP**

# Independence from Double Negation Translation

- $\vee^e$, $\wedge^e$, $\propto$, $\bot$ and $\top$ are now first-class **green** connectives and constants.
- $A^\bot$ is now De Morgan negation, defined by "dualities:"
  $\vee^e/\wedge$, $\wedge^e/\vee$, $\propto/\supset$, $\bot/1$, $\top/0$,
- "dual atoms" $a/a^\bot$; Formulas are in negation normal form.
  **If $A \vdash_\circ B$ is provable, then $B^\bot \vdash_\bullet A^\bot$ is provable.**
- Reclassification of some formulas:
  1 and $R \supset \bot$ are now **red**. $(R \supset \bot)^\bot = R \propto 1$
- Every green formula is of the form $R^\bot$ for some red $R$.
  Given $A$ and $A^\bot$, one is red, the other is green.

# Kripke Semantics

**Hybrid Model** **(Propositional Case):** $\langle \mathbf{W}, \preceq, \mathbf{C}, \models \rangle$
Requirements and definitions:

- $\preceq$ is a transitive, reflexive ordering on non-empty set **W** of "possible worlds."
- $\models$ is a monotonic relation between elements of **W** and sets of atomic formulas.
- $\mathbf{C} \subseteq \mathbf{W}$ ("classical worlds")
- $\triangle_{\mathbf{u}} = \{\mathbf{k} \mid \mathbf{k} \in \mathbf{C} \text{ and } \mathbf{u} \preceq \mathbf{k}\}$ ("classical cover" of **u**)
- required: $\triangle_{\mathbf{k}} = \{\mathbf{k}\}$ for all $\mathbf{k} \in \mathbf{C}$. (for propositional models)
- if $\triangle_{\mathbf{u}} = \emptyset$ then **u** is **imaginary**.

*Every Kripke Model for IL is immediately a Hybrid Model, with a more structured interpretation of possible worlds.*

# Rules of $\models$

for $\mathbf{u}, \mathbf{v} \in \mathbf{W}$; $\mathbf{c}, \mathbf{k} \in \mathbf{C}$, green $E$:

- $\mathbf{u} \models 1$ and $\mathbf{u} \not\models 0$
- $\mathbf{u} \models A \vee B$ iff $\mathbf{u} \models A$ or $\mathbf{u} \models B$
- $\mathbf{u} \models A \wedge B$ iff $\mathbf{u} \models A$ and $\mathbf{u} \models B$
- $\mathbf{u} \models A \supset B$ iff for all $\mathbf{v} \succeq \mathbf{u}$, $\mathbf{v} \models A$ implies $\mathbf{v} \models B$

- **$\mathbf{u} \models E$ iff for all $\mathbf{k} \in \triangle_{\mathbf{u}}$, $\mathbf{k} \models E$**
- **$\mathbf{c} \models E$ iff $\mathbf{c} \not\models E^{\perp}$**

E.g., $\mathbf{c} \models A \propto B$ iff $\mathbf{c} \not\models A \supset B^{\perp}$ iff for some $\mathbf{v} \succeq c$, $\mathbf{v} \models A$ and $\mathbf{v} \not\models B^{\perp}$.
Monotonicity preserved by condition $\triangle_{\mathbf{c}} = \{\mathbf{c}\}$.

- **If $\triangle_{\mathbf{u}} = \emptyset$, then $\mathbf{u} \models E$ for all green $E$.**
- **$\mathbf{u} \models A \vee^{e} A^{\perp}$**

# Important Countermodels

$$s_1 : \{a, a^\perp\} \qquad\qquad s_2 : \{a^\perp\}$$
$$\nwarrow \quad \nearrow$$
$$k : \{a^\perp\}$$

**shows that $a \vee^e \sim a$ and $\sim a \vee^e \sim\sim a$ are not valid**

***shows that intuitionistic implication does not collapse***

$$k : \{p, q\}$$
$$\uparrow$$
$$s : \{\}$$

**shows that $(p \wedge^e q) \supset p$, $(p \vee^e q) \supset (p \vee q)$, etc... are not valid:**

$$\frac{P \vdash R}{P \wedge^e Q \vdash R} \wedge L \qquad \frac{P \vdash R \quad Q \vdash R}{P \vee^e Q \vdash R} \vee L \qquad \frac{P \wedge^e Q}{P} \wedge E$$

**... are not valid inference rules; some *device* needed.**

# Semantics and Cut Admissibility

**LP is sound/complete by Hintikka-Henkin constructions**

**Some admissible cuts guaranteed by semantics:**

$$\frac{\Gamma \vdash_{\circ} A \quad A, \Gamma' \vdash_{\circ} B}{\Gamma\Gamma' \vdash_{\circ} B}\ \mathit{Cut} \qquad \frac{A, \Gamma \vdash_{\bullet} \Theta \quad A^{\perp}, \Gamma' \vdash_{\bullet} \Theta'}{\Gamma\Gamma' \vdash_{\bullet} \Theta\Theta'}\ \mathit{cut}_{\bullet} \qquad \frac{\Gamma \vdash_{\circ} A \quad \Gamma' \vdash_{\circ} A^{\perp}}{\Gamma\Gamma' \vdash_{\bullet}}\ \mathit{cut}_{\perp}$$

**A non-admissible cut:**

$$\frac{\Gamma \vdash_{\bullet} P \quad P, \Gamma' \vdash_{\circ} Q}{\Gamma\Gamma' \vdash_{\circ} Q}\ \textbf{bad cut}$$

when $P$, $Q$ are **red**.

$$\begin{array}{c} k : \{P, Q\} \\ \uparrow \\ s : \{\} \end{array}$$

# Procedural Cut Elimination

$$
\dfrac{
\dfrac{
\dfrac{A^\perp, B^\perp, \Gamma \vdash_\bullet}{\Gamma \vdash_\bullet A, B}\,\mathit{Store} \times 2
}{\Gamma \vdash_\bullet A \vee^e B}\,\vee^e R
\qquad
\dfrac{
\dfrac{A \vee^e B, \Gamma' \vdash_\circ A^\perp \quad A \vee^e B, \Gamma' \vdash_\circ B^\perp}{A \vee^e B, \Gamma' \vdash_\circ A^\perp \wedge B^\perp}\,\wedge R
}{A \vee^e B, \Gamma' \vdash_\bullet}\,\mathit{Load}
}{\Gamma\Gamma' \vdash_\bullet}\,\mathit{cut}
$$

**Reduces to:**

$$
\dfrac{
\dfrac{
\dfrac{\Gamma \vdash_\bullet A \vee^e B}{\Gamma \vdash_\circ A \vee^e B}
\quad A \vee^e B, \Gamma' \vdash_\circ B^\perp
}{\Gamma\Gamma' \vdash_\circ B^\perp}\,\mathit{cut}
\qquad
\dfrac{
\dfrac{
\dfrac{\Gamma \vdash_\bullet A \vee^e B}{\Gamma \vdash_\circ A \vee^e B}
\quad A \vee^e B, \Gamma' \vdash_\circ A^\perp
}{\Gamma\Gamma' \vdash_\circ A^\perp}\,\mathit{cut}
\quad A^\perp, B^\perp, \Gamma \vdash_\bullet
}{B^\perp, \Gamma\Gamma' \vdash_\bullet}\,\mathit{cut}
}{\Gamma\Gamma' \vdash_\bullet}\,\mathit{cut}
$$

# Let's Be Naive ...

$$\frac{A, \Gamma \vdash_\circ R}{A \wedge^e B, \Gamma \vdash_\circ R}\ \textit{naive-}\wedge^e L$$

Try to reduce the following cut:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash_\bullet A \quad \Gamma \vdash_\bullet B}{\Gamma \vdash_\bullet A \wedge^e B}\ \wedge^e R}{\Gamma \vdash_\circ A \wedge^e B}\ \textit{Signal} \qquad \dfrac{A, \Gamma' \vdash_\circ R}{A \wedge^e B, \Gamma' \vdash_\circ R}\ \textit{naive-}\wedge^e L}{\Gamma\Gamma' \vdash_\circ R}\ \textit{Cut}$$

would require

$$\frac{\Gamma \vdash_\bullet A \qquad A, \Gamma\Gamma' \vdash_\circ R}{\Gamma\Gamma' \vdash_\circ R}\ \textbf{bad cut}$$

**Violates LP Invariant: no green introduction rules in $\vdash_\circ$ mode**

# Alternative Proof System LPM

## Right-Red Rules

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee R \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge R \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B, \Delta}\ \supset R \qquad \frac{}{\Gamma \vdash 1, \Delta}$$

## Left-Red Rules

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\ \vee L \qquad \frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ \wedge L \qquad \frac{A \supset B, \Gamma \vdash A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta}\ \supset L$$

## Left-Green Rules

$$\frac{A, \Gamma \vdash \quad B, \Gamma \vdash}{A \vee^e B, \Gamma \vdash}\ \vee^e L \qquad \frac{A, B, \Gamma \vdash}{A \wedge^e B, \Gamma \vdash}\ \wedge^e L \qquad \frac{A, \Gamma \vdash B^\perp}{A \propto B, \Gamma \vdash}\ \propto L \qquad \frac{}{\perp, \Gamma \vdash}\ \perp L$$

## The Lift Rule and Identity

$$\frac{E^\perp, \Gamma \vdash}{\Gamma \vdash E, \Delta}\ \mathit{Lift} \qquad \frac{}{a, \Gamma \vdash a, \Delta}\ I_r \qquad \frac{}{a, a^\perp, \Gamma \vdash}\ I_\ell \qquad \frac{}{0, \Gamma \vdash \Delta}\ 0L$$

$E$ is a green formula and $a$ is an atomic formula

# Some Properties of PIL

- $A \vee^e \neg A$ is valid/provable (LEM)
- if $A \vee B$ provable, either $A$ or $B$ provable (Disjunction Prop.)
- $A \propto 1 \;\equiv\; A \vee^e \bot \;\equiv\; \neg\neg A$
- Classical and intuitionistic connectives can mix freely:
  In $A \vee^e (B \supset C)$, $\supset$ *does not collapse*

  **But limitations still exist...**

- Define **classical implication**: $A \Rightarrow B = A^\perp \vee^e B$:
  Can prove

$$((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$$

$$((P \Rightarrow Q) \supset P) \Rightarrow P$$

$$((P \supset \bot) \supset P) \Rightarrow P$$

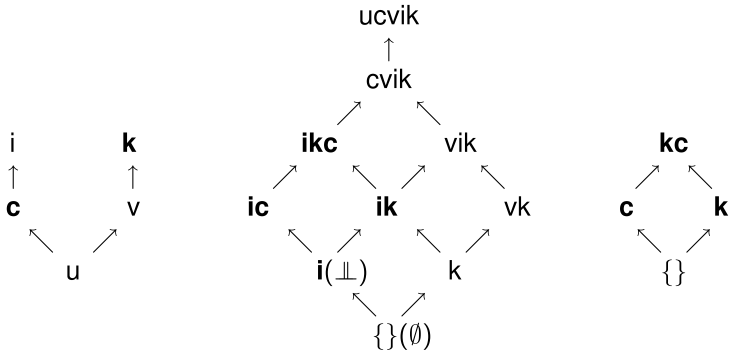
  **but not**

$$((P \supset \bot) \supset P) \supset P$$

  And the outermost $\supset$ is most important.

# Algebraic Perspective

$$\perp\!\!\!\perp \;=\; \{\mathbf{u} \in \mathbf{W} : \; \mathbf{u} \models \perp\} \;=\; \{\mathbf{u} \in \mathbf{W} : \; \triangle_{\mathbf{u}} = \emptyset\}$$




Kripke Frame, $\mathbf{C} = \{\mathbf{c}, \mathbf{k}\}$ Heyting Algebra with $\perp\!\!\!\perp$ Boolean Algebra $2^{\mathbf{C}}$

**Embedded Algebra** $= \{K \cup \perp\!\!\!\perp : \; K \subseteq \mathbf{C}\}$

# Interpretation of formulas

- $h(1) = h(\top) = \mathbf{W}$; $\quad h(\bot) = \perp\!\!\!\perp$
- $h(A \vee B) = h(A) \sqcup h(B), \quad h(A \wedge B) = h(A) \sqcap h(B)$
- $h(A \supset B) = h(A) \rightarrow h(B)$.
- $h(R^{\bot}) \;=\; h(R) \rightarrow \perp\!\!\!\perp$ for all green $R^{\bot}$.

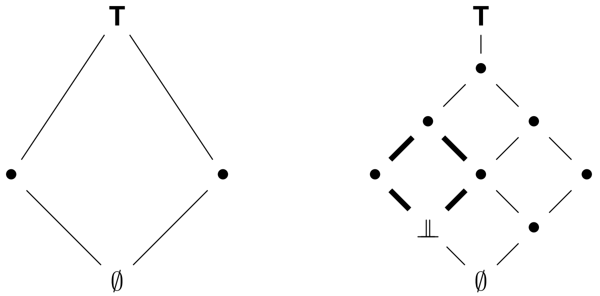
Top of embedded boolean algebra = $\mathbf{C} \cup \perp\!\!\!\perp$.

(Alternatively, let $\mathbf{1} = \mathbf{C} \cup \perp\!\!\!\perp$, change $\Gamma \vdash_{\circ} 1$ to $\Gamma \vdash_{\bullet} 1$)

Define secondary interpretation $h'(A) = (h(\neg\neg A) \cap \mathbf{C}) \cup \perp\!\!\!\perp$:

- $h'(A \wedge B) = h'(A \wedge^{e} B) = h'(A) \cap h'(B)$
- $h'(A \vee B) = h'(A \vee^{e} B) = h'(A) \cup h'(B)$
- $\overline{h'(A)} = h'(A^{\bot})$; $\quad \overline{X}$ is boolean complement in embedded algebra.
- **But** $h'(A \supset B) \neq h'(A) \rightarrow h'(B)$
- **$h'(E) = \mathbf{C} \cup \perp\!\!\!\perp$ iff $h(E) = \mathbf{T}$ for green $E$.**

# Black Hole ($\sim\sim$) versus Worm Hole ($\neg\neg$)




**Black Hole: all points $A \to \emptyset$ (or $(A \to \emptyset) \to \emptyset$) (Glivenko 1929)**

*Not closed under $\vee$, closed under $\to$*
$(A \to \emptyset) \to (B \to \emptyset) \;\equiv\; ((A \to \emptyset) \wedge B) \to \emptyset.$ **No escape!**

**Worm Hole: all points $(A \cap \mathbf{C}) \cup \perp\!\!\!\perp$ (based on $\neg\neg A$)**

*Closed under $\vee$, not closed under $\to$*

# Semantic Alternatives; Conclusions

- Require $\mathbf{C} \neq \emptyset$: so $\perp\!\!\!\perp \neq \mathbf{T}$
  But $\perp$ is nolonger just an atom.
- Add **red** constant $\lozenge = \mathbf{C} \cup \perp\!\!\!\perp$: $\lozenge \approx ?1$; $\mathbf{k} \models \lozenge$ iff $\mathbf{k} \in \mathbf{C}$.

$$\frac{}{\Gamma \vdash_{\bullet} \lozenge}\ \lozenge R$$

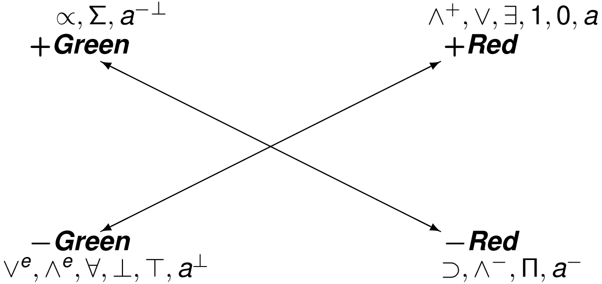
- Extend to first order quantifiers: lose property $\triangle_{\mathbf{c}} = \{\mathbf{c}\}$
- Let $\perp\!\!\!\perp$ be the second-largest element: ICL
  $A \vee^{-} \neg A$ valid without a different version of disjunction.

# Summary

- **Can a double-negation translation allow us to combine classical logic with intuitionistic logic?**
- **Yes,** polarize the doubly-negated formulas; then **focus.**
- Derive sequent calculus LP with two modes $\vdash_\circ$ and $\vdash_\bullet$; satisfies cut-elimination
- Intuitionistic implication does not collapse in PIL.
- $A \equiv B$ intuitionistically if $A \vdash_\circ B$ and $B \vdash_\circ A$;
  implies $B^\perp \vdash_\bullet A^\perp$ and $A^\perp \vdash_\bullet B^\perp$
- Semantics completes the lifting of labels into connectives;
  Defines new logic.
- No need to involve linear logic.
  $?A^\perp \parr B$ $(!A \multimap B)$ is properly linear ($B$ is "neutral").
  No neutrals needed; combination can occur within intutionistic logic, **with focusing and enriched semantics.**

# LPF: Focused LP

**Separate positive/negative from red/green polarization**



# Can we *cross-focus* between **+Green** and **+Red?**

- **+** to **+**: OK (focusing in LJF).
- **+** to **+**: OK; $A \propto (B \propto C) \; = \; \neg(A \supset (B \supset C^{\perp}))$.
- **+** to **+**: OK; $(A \vee B) \propto C \; = \; \neg((A \vee B) \supset C^{\perp})$
- **+** to **+**: **Not a chance!** $A \vee (B \propto C) \; = \; A \vee \neg(B \supset C^{\perp})$.
  Pattern is $+ - +$. In linear logic, $!A \oplus !?(!B \otimes C)$

Need **two layers of focusing** with **lateral transition rules**.

$\Uparrow^{\bullet}/\Downarrow^{\bullet}$ along -/+ axis.
$\Uparrow^{\circ}/\Downarrow^{\circ}$ along -/+ axis.

$\vdash_{\circ}$ corresponds to $\Uparrow^{\circ}, \Downarrow^{\bullet}$.
$\vdash_{\bullet}$ corresponds to $\Uparrow^{\bullet}, \Downarrow^{\circ}$.

# LPF (one sided version)

**Structural/Reaction Rules**

**Lateral Reactions**

$$\frac{\vdash \Gamma : \Delta \Uparrow^{\bullet} \Upsilon}{\vdash \Gamma : \Delta \Uparrow^{\circ} \Upsilon}\ L\Uparrow \qquad \frac{\vdash \Gamma : \Downarrow^{\bullet} R}{\vdash \Gamma : \Downarrow^{\circ} R}\ L\Downarrow$$

**Negative Reactions**

$$\frac{\vdash \Gamma : R \Uparrow^{\circ} \Theta}{\vdash \Gamma : \Uparrow^{\circ} R, \Theta}\ R_1\Uparrow \qquad \frac{\vdash D, \Gamma : \Delta \Uparrow^{\bullet} \Theta}{\vdash \Gamma : \Delta \Uparrow^{\bullet} D, \Theta}\ R_2\Uparrow \qquad \frac{\vdash \Gamma : \Downarrow^{\circ} S}{\vdash \Gamma : S \Uparrow^{n}}\ D_1 \qquad \frac{\vdash T, \Gamma : \Delta \Downarrow^{\circ} T}{\vdash T, \Gamma : \Delta \Uparrow^{n}}\ D_2$$

**Positive Reactions**

$$\frac{\vdash \Gamma : \Delta \Uparrow^{\bullet} N}{\vdash \Gamma : \Delta \Downarrow^{\circ} N}\ R_1\Downarrow \qquad \frac{\vdash \Gamma : \Uparrow^{\circ} M}{\vdash \Gamma : \Downarrow^{\bullet} M}\ R_2\Downarrow \qquad \frac{}{\vdash \Gamma : a^{\perp} \Downarrow^{n} a}\ I_1 \qquad \frac{}{\vdash a^{\perp}, \Gamma : \Downarrow^{n} a}\ I_2$$

$\Upsilon$ contains only green formulas; $R$: red formula; $D$: positive formula or negative green literal; $S$: positive red formula; $T$: positive formula; $N$: negative green formula; $M$: negative formula; $a$, positive atom.

# LPF Introduction Rules

## Constants

$$\frac{\vdash \Gamma : \Delta \Uparrow^{\bullet} \Theta}{\vdash \Gamma : \Delta \Uparrow^{\bullet} \bot, \Theta}\ \bot \qquad \frac{}{\vdash \Gamma : \Delta \Uparrow^{\bullet} \top, \Theta}\ \top \qquad \frac{}{\vdash \Gamma : \Downarrow^{\bullet} 1}\ 1$$

## Negative Connectives

$$\frac{\vdash \Gamma : \Delta \Uparrow^{\bullet} A, B, \Theta}{\vdash \Gamma : \Delta \Uparrow^{\bullet} A \vee^{e} B, \Theta}\ \vee^{e} \qquad \frac{\vdash \Gamma : \Delta \Uparrow^{\bullet} A, \Theta \quad \vdash \Gamma : \Delta \Uparrow^{\bullet} B, \Theta}{\vdash \Gamma : \Delta \Uparrow^{\bullet} A \wedge^{e} B, \Theta}\ \wedge^{e} \qquad \frac{\vdash \Gamma : \Delta \Uparrow^{\bullet} A, \Theta}{\vdash \Gamma : \Delta \Uparrow^{\bullet} \forall x.A}$$

$$\frac{\vdash \Gamma : \Uparrow^{\circ} A, \Upsilon}{\vdash \Gamma : \Uparrow^{\circ} \Pi x.A, \Upsilon}\ \Pi \qquad \frac{\vdash \Gamma : \Uparrow^{\circ} A, \Upsilon \quad \vdash \Gamma : \Uparrow^{\circ} B, \Upsilon}{\vdash \Gamma : \Uparrow^{\circ} A \wedge^{-} B, \Upsilon}\ \wedge^{-} \qquad \frac{\vdash \Gamma : \Uparrow^{\circ} B, A^{\perp}, \Upsilon}{\vdash \Gamma : \Uparrow^{\circ} A \supset B, \Upsilon}\ \supset R$$

$x$ is not free in $\Gamma, \Delta, \Theta$; $\Upsilon$ contains only green formulas

## Positive Connectives

$$\frac{\vdash \Gamma : \Downarrow^{\bullet} A \quad \vdash \Gamma : \Downarrow^{\bullet} B}{\vdash \Gamma : \Downarrow^{\bullet} A \wedge^{+} B}\ \wedge^{+} \qquad \frac{\vdash \Gamma : \Downarrow^{\bullet} A_i}{\vdash \Gamma : \Downarrow^{\bullet} A_1 \vee^{+} A_2}\ \vee^{+} \qquad \frac{\vdash \Gamma : \Downarrow^{\bullet} A[t/y]}{\vdash \Gamma : \Downarrow^{\bullet} \exists y.A}\ \exists$$

$$\frac{\vdash \Gamma : \Delta \Downarrow^{\circ} A[t/y]}{\vdash \Gamma : \Delta \Downarrow^{\circ} \Sigma y.A}\ \Sigma \qquad \frac{\vdash \Gamma : \Downarrow^{\bullet} D \quad \vdash \Gamma : \Delta \Downarrow^{\circ} B}{\vdash \Gamma : \Delta \Downarrow^{\circ} D \propto B}\ \propto\ (\supset L)$$